EXTRACT FROM THE ...

(Zeitschrift für angewandte Mathematik und Mechanik

No. 4. Vol. 12 August 1932).

Phenomena associated with Impacts and Sliding on liquid surfaces.

Herbert Wagner.

The matter dealt with in the present paper has a direct bearing on the taking-off and alighting of seaplanes. The liquid is assumed as being non-viscous and incompressible and the acceleration due to gravity is neglected. The following considerations therefore apply the more exactly the smaller the body and the more quickly it moves. In the case of prolonged, and primarily, of uniform processes, the arguments hold only in the close neighbourhood of the body.

The investigations deal first with the boundary conditions with regard to the discontinuities which appear at the free surface. The various movements of the liquid may be divided into two classes, which may be usefully termed impact and "sliding " movements x according to the character of these discontinuities.

---

xFoot Note. "Sliding movements" are the same as "flowing away" but in the present connection the latter term does not appear exactly suitable.

In the case of impacts (Fig. 1, 2, 23 top) the free surface, during the period of time under consideration, consists of the same liquid particles. If a contour exists between the free surface and the body, this will be formed along its whole length of "peaks" (edges of liquid wedges) fig. 7. These peaks frequently degenerate to splashes which may be interpreted as the expression of the momentum "destroyed" by the impact. Impact processes are determined uniquely by the initial motion of the liquid and the course of the movement of the body.

In "sliding" processes (figs. 12, 26) there are regions along the contour between the surfaces of the body and of the liquid (e.g. sliding edges), along which new liquid particles come to the surface. For the determination of sliding processes, in addition to the data relative to the initial movement of the liquid and of the motion of the body, certain assumptions are necessary with regard to the position of the point of diversion of the flow from the surface of the body. If, within the region of the flow there is a projecting edge, and if infinitely great negative pressure ? is to be prevented at this position, the diversion must take place along this edge x

x Foot Note. The process illustrated in fig. 23 (centre) owing to the position of diversion C. ("Sliding edge") is also a slide. Its general treatment as an impact is an approximation, which gives accurate results only in the limiting case of infinitely small splash thickness.

The processes of sliding may be compared with the motion of fluid past a body from which a sheet of vorticity breaks away. The determination of the tangential velocity of the fresh liquid particles arriving at the surface is analogous to the determination of the strength of vorticity of the newly-formed vortex filaments. A detailed analysis will be made of the motion of bodies with infinitely flat bottom on the surface of the liquid (fig. 12). In the earlier method of treatment of the problem by means of pressure points [xx] account was taken of the acceleration due to gravity and an explanation obtained of wave form. Owing to neglect of the discontinuity (splash) at the forward edge however, the method rendered it possible only in special cases, to derive an interpretation of the flow processes near the body and to obtain a correct calculation of the resistance. In the present paper it is shown that, in the limiting case under consideration, the uniform or variable sliding or impact motions of a flat bottom on the free surface are determined by the "equivalent aerofoil motions". Except in the region of the splash, the same pressure exists at the pressure surface of the bottom as at the pressure side of the equivalent aerofoil; instead of the suction

---

xx Lamb - Hydrodynamics. pp. 440 et seq. (esp.fig. p.1454) and pp. 437 et seq.)

which occurs in the case of the aerofoil, a splash is thrown off at the edge of the pressure surface, the energy of which corresponds to the increase in drag, resulting from the removal of the suction force. Whereas in the case of the aerofoil the drag (induced drag) depends only on the lift, in the case of the bottom the additional resistance due to the loss of the suction varies even with the same buoyancy for different shapes of bottom. A few examples will show how the boundary condition for the contour of the pressure surface (rise of the liquid particles equal to the height of the bottom, cf. para. 10.) may be fulfilled.

In addition, equations are found for two dimensional liquid motion with free surface when a centre of similitude is present. Examples of such motion are : The two-dimensional problem of the penetration of a wedge at constant speed V_s into a liquid with initially a plane or wedge-shaped surface (figs. 2, 26, 29) and the axially symmetrical problem of the penetration of a cone at constant speed into a liquid with initially a plane or conical surface.

Finally the case, which, as far as the writer is aware, has not hitherto been dealt with, of the uniform glide of a flat plate is calculated (two-dimensional problem, finite angle of incidence).

<u>Symbols.</u> Scalar, in Latin or Greek letters; vectors in German letters :

t	=	time.
φ	=	velocity potential.
$\mathfrak{v} = \operatorname{grad} \varphi$, v	=	velocity of the liquid.
D/Dt	=	Stokes' operator (for the substantial variation).
$\mathfrak{n}$	=	Unit vector perpendicular to the surface.
$\mathfrak{v}_n$, $v_n = \mathfrak{n}_n \cdot \mathfrak{v}$	=	normal } Components of the velocity
$\mathfrak{v}_t$, v_t	=	tangential } of the liquid at the surface.
$\mathfrak{V}$ V	=	translational velocity of a rigid body.
V_n	=	normal velocity of the contour of a body (generally varying its form).
ρ	=	specific density of the liquid.

Further, for the two-diminsional problem of fluid notion:

α	=	angle of inclination of the surface (in general with reference to the initial plane surface).
s	=	Length of arc of the surface.
$\mathfrak{t}$	=	Unit vector in the direction of the tangent to the surface.
ψ	=	flow function.

$z = x + i\,y$ = complex co-ordinate.

$w = \varphi + i\,\psi$ = complex flow function.

$\dfrac{dw}{dz} = v_x - i\,v_y$ = complex velocity.

I. FUNDAMENTAL PRINCIPLES.

1. Boundary condition. The interpretion of the geometrical boundary condition in the sense employed below is, that at the surface of the liquid (at the body and at the free surface) the normal velocity v_n of the liquid is equal to the translational velocity of the surface perpendicular to itself *. At the free surface this is frequently replaced in the two-dimensional problem by its differential form ** (cf. fig. 3).

$$\frac{D\alpha}{Dt} = v_n \frac{\partial V}{\partial s} \qquad (1).$$

but the original condition itself must then be satisfied at one point.

---

* Foot Note. cf. Lamb. loc. cit.

** Use is made of $\partial/\partial s$, because here and in all subsequent data on the run of the velocity at the surface, the velocity of the liquid particles of the surface itself only will be considered.

At the free surface we have, in addition, the dynamic boundary condition. Since the free surface is a surface of constant pressure, the pressure gradient and consequently the acceleration $\frac{D\mathfrak{w}}{Dt}$ is perpendicular to the free surface. Hence for the two-dimensional problem.

$$\mathfrak{n}_t \frac{D\mathfrak{w}}{Dt} = 0 \qquad (2).$$

or with

$$\frac{D\mathfrak{n}_t}{Dt} = \mathfrak{n}_n \frac{D\alpha}{Dt}$$

$$\frac{Dv_t}{Dt} = \frac{D\mathfrak{n}_t\mathfrak{w}}{Dt} = \mathfrak{w}\frac{D\mathfrak{n}_t}{Dt} = v_n \frac{D\alpha}{Dt} \qquad (3).$$

Combining the dynamic boundary condition equation (3) with the geometrical (1) we get :-

$$\frac{Dv_t}{Dt} = v_n \mathfrak{n}_n \frac{\partial \mathfrak{w}}{\partial s} \qquad (4).$$

In the two-dimensional problem also, $D\mathfrak{n}_n / Dt$, owing to the geometrical boundary condition, is determined by $\mathfrak{w}$ at the free surface.

$$\frac{D\mathfrak{n}_n}{Dt} = \left[\mathfrak{n}_n \, [\mathfrak{n}_n \, \mathrm{grad}]\right] \cdot \mathfrak{n}_n \mathfrak{w} \qquad (5).$$

The dynamic boundary condition

$$\left[\mathfrak{n}_n \frac{D\mathfrak{w}}{Dt} \right] = 0$$ is expressed in the form.

$$\frac{D\mathfrak{w}_t}{Dt} = \frac{D}{Dt}\left[\mathfrak{n}_n[\mathfrak{n}_n \mathfrak{w}]\right] = \mathfrak{n}_n \cdot \mathfrak{w} \frac{D\mathfrak{n}_n}{Dt} + \mathfrak{n}_n \mathfrak{w} \cdot \frac{D\mathfrak{n}_n}{Dt} \qquad (6).$$

From equations (4) or (5) and (6) we obtain:

Theorem I: If, for a given moment of time the velocity $\mathfrak{w}$ together with its directional derivative (in the direction of the surface) for all liquid particles of the free surface are known, then the variation in time of the tangential velocity $\mathfrak{w}_t$ for these particles is also known.

2. Boundary condition with discontinuous curvature.

Preliminary consideration: For a two-dimensional problem of fluid motion take the following as hypothesis (fig.4): On a portion s_1 of the free surface, let the form of the surface move parallel to itself at a velocity $\mathfrak{w}_1$ invariable in time. All liquid particles situated at the beginning B of s_1 should, at this position, have the same velocity $\mathfrak{w}_B$.

Owing to the geometrical boundary condition, along s_1 the "relative" velocity of the liquid particles is $\mathfrak{n}_t U = \mathfrak{w} - \mathfrak{w}_1$, where U is the amount of the relative velocity. By putting $\mathfrak{w}$ from this into the dynamic boundary equation (2) we get $DU / Dt = 0$, that is, the value U is constant for every liquid particle in its movement along s_a. Since at point B, $U = |w_B - w_1|$ is of the same value for all the liquid particles, it follows from the hypothesis that U is constant along s_a.

Boundary transition.

This consideration will be applied to an (infinitely) small region $s_a \longrightarrow 0$, along which the inclination of the free surface in general differs by a finite angle (fig.5). At the surface adjacent to B outside s_a, let the curvature of the surface and the velocity field vary continuously and continuously in time. The displacement of this region s_a with a continuously variable velocity w_1, is given by the foregoing consideration: if the form of the surface s_a varies continuously on a (finite) path W [x], then along s_a at any moment U differs from a constant value only by ΔU, $\frac{\Delta U}{U}$ being small as $\frac{s_a}{W}$ Or briefly:

---

[x] Foot Note. i.e. the surface increased in the scale I/s_a should retain its character (at least in part) of a smooth curve. It should also be possible to select on s_a an infinite sequence of points (interval Δ) so that on the path W every Δ experiences a rotation, continuously variable in time of a (generally) finite angle.

<u>Theorem 2: For every small region $s_3 \longrightarrow 0$ of the free surface with continuous or dis-continuous curvature, the form of which travels "approximately unchanged" at a velocity w_1, with U = const, $w - w_1 \to n_t U$;</u>

<u>i.e., the boundary condition at the free surface, for the relative velocity field $w - w_1$ is the same as with a uniform flow.</u>

With corresponding conditions of continuity, this consideration may be formulated also for the three-dimensional case. It is obtained with the notation established for the two-dimensional problem $w = w_1 + n_t U + \vec{c}$, where $\vec{c}$ denotes a constant velocity along s_1 in the direction of the edge.

3. <u>Flow in the region of discontinuous curvature.</u>

<u>Angles:</u> Let the discontinuity s_3 (fig. 3) and the region of fluid G_1 be enveloped by the infinitely small circle r_1. The region of fluid G_2 lies between the circle r_1 and the larger (e.g. concentric)(but still infinitely small) circle r_2. In accordance with theorem 2, w_F is constant along F_F, and w_G along G_1. In accordance with the theorem [x] of the theory of functions:

---

[x] Foot Note. Hurwitz - Courant. Funktionentheorie Springer 1929. p. 375?

"If, in a region (G_g) the boundary value of an analytical function at an arc of a smooth curve is zero, it is identically zero"[x], the relation $w - w_B = 0$ must hold in the region G_2. Hence it follows that $w_C = w_B$ which, in accordance with theorem 2 is consistent only with $U = 0$ hence with $w = w_1$. Consequently $w = w_1$ along the whole contour of G_1, hence also at G_s itself.

<u>Theorem 3:</u> <u>In a re-entrant or projecting dis-continuity of the free surface, situated at a distance of at least r_2 from other dis-continuities or from the surface of the body, the velocity of the fluid is equal to the velocity w_1 of the dis-continuity.</u>

<u>Peak of the Splash:</u> Using the same theorem of the theory of functions, it may be shown for the case (fig. 7), when along $s_1 / 2$ the contour of the body and of the free surface is continuously curved and when the normal velocity V_n at the body is continuous and continuously variable in time, that:

<u>Theorem 4:</u> <u>If the free surface forms a peak with the surface of the body, the velocity of the fluid in the peak is equal to the velocity of the peak.</u>

<u>Root of the Splash.</u> Consider the case of a

---

x c.f. Lamb, loc. cit. p. 48.

dis-continuity in the neighbourhood of the body. Using the method of Schwarz - Christoffel [x] we find for the flow (fig. 8).

$$\frac{dw}{dt} = \frac{\tau + 1}{\tau} \quad \text{and} \quad d \ln \frac{dz}{dw} = - i \frac{d\tau}{(\tau + 1)\sqrt{\tau}}$$

where τ is the complex auxiliary plane. If $-U$ is the velocity at the free surface and δ the thickness of the splash we get :

$$w = - U \frac{\delta}{\pi} \left(1 + \ln \tau + \tau \right) \qquad (7).$$

$$z = \frac{\delta}{\pi} \left(\ln \frac{-1}{\tau} + 4 i \sqrt{\tau} + \tau + 5 \right) \qquad (8).$$

The free surface is given by

$$\pi \frac{x}{\delta} = 5 + \frac{\pi^2}{16} \left(\frac{y}{\delta} - 1 \right)^2 - \ln \left[\frac{\pi^2}{16} \left(\frac{y}{\delta} - 1 \right)^2 \right] \qquad (9)$$

At the bottom (negative real τ) the pressure of the fluid (fig.9) is

$$p = 2 \rho U^2 \frac{\sqrt{-\tau}}{(1 + \sqrt{-\tau})^2} \qquad (10).$$

---

x Lamb. loc cit. p. 100.

The maximum pressure $p_{max} = \frac{1}{2} \rho U^2$ occurs at the position $\tau = -1$, hence $x = 0$. The pressure P_w on the portion of the bottom between x and $x = + \infty$ is $P_w = \frac{4}{\pi} \rho \delta U^2 \sqrt{-\tau}$. For $\tau \longrightarrow - \infty$, according to equation (8) $\frac{z}{\tau} \longrightarrow \frac{\delta}{\pi}$; hence

$$\text{for} \quad x \longrightarrow - \infty, \quad P_w \longrightarrow 4 \rho U^2 \sqrt{\frac{\delta}{\pi}} \sqrt{-x} \tag{11}$$

If a constant velocity v_1 is superposed on the flow (fig. 8), a flow non-stationary, in the strict sense of the word, is produced without any change taking place in the pressure distribution. If, in particular a velocity $v_1 = U$ in the direction of the bottom, is superposed (fig. 9), and hence according to equation 7,

$$w_1 = v_1 \left[\frac{-\delta}{\pi} (1 + \ln \tau + \tau) + z \right] \tag{12}$$

the tangential velocity at the free surface at a sufficient distance from the splash will be zero. For z and τ of such value that $\ln \tau$ is negligible in relation to $\sqrt{\tau}$ equation (12), taking into account equation (8) is transformed into the form

$$w_1 \longrightarrow 4 v_1 \sqrt{\frac{\delta}{\pi}} \cdot \sqrt{-z} \tag{13}$$

A flow similar t... at described here will occur in a small region in every case of impact or glide.

Gliding edge. Let us apply theorem 2 to the free surface behind a gliding edge. Assuming continuous curvature of the surface of the body at this point, the relative flow in a region of (infinitely) small z (fig.10) is given by:

$$d \ln \frac{dz}{dw} = \frac{i\,dw}{\pm\sqrt{w}} \qquad (14).$$

hence:

$$\frac{z}{A} = 2\int_0^w e^{\pm i\sqrt{\frac{w}{AU}}}\, d\frac{w}{AU} \qquad (15).$$

A is a constant determining the scale, U is the relative tangential velocity.

4. Uniqueness.

Consider the case of an impact, in which at the time t = 0, the form of the free surface and the surface of the body and the velocity field of the fluid are given. Let the velocity of the moving boundary of the surface of the body (V_n) be given for $t \geq 0$. If in the course of time no new fluid particles reach the surface (impact), the course of the flow in time is uniquely determined by these data, since:

With given velocity field (hence also v_n) at the time t, the form of the surface at the time t + dt is uniquely determinable. At the surface of the body, according to hypothesis V_n is also given at the time $t + dt$. At the free surface, at the time $t + dt$ (theorem 1): φ_t and hence with simply - connected surface, φ also (except for an insignificant constant) are uniquely determinable. By means of these data V_n and φ, the velocity field of the fluid at the time $t + dt$ x and hence also v_n at the free surface may be uniquely determined and so on, for all subsequent dt.

Only the discontinuities (besides sliding edges) defined earlier however may appear. This proof may be applied for the actual calculation of general cases (even when highly involved) of non-stationary impacts.

In the case of sliding, as shown in fig. 10, it is immaterial as far as the flow in the neighbourhood of the point of diversion is concerned, whether the latter is a sliding edge or whether the bottom aft of the point of diversion extends in a continuous curve. If in the latter case the variation in time of the position of the point of diversion is given, the two

---

x Lamb loc. cit. p.46.

problems are equivalent. [x]

The writer considers that beyond doubt a slide may be uniquely determined if particulars of the surface and the velocity field of the fluid at the time t = 0,

the run of the contour of the body and the position of the point of diversion for t ≧ 0 are given. In the proof used for impacts however the following difficulty is encountered.

During the time dt, a new portion of the free surface has formed, the tangential velocity of which is not

---

[x] Foot Note. The selection of the point of diversion in the case of a continuously curved bottom is subject to the limiting condition, that the flow is diverted outwards from the surface of the bottom. This, according to the flow in fig. 10, is associated with a rise of pressure in the direction of the velocity. Obviously the point of diversion should be so selected that this rise in pressure is avoided. This may be still just obtained by the possible condition of continuous curvature at this position as a limiting case. With the glide of flying boats (slightly curved bottom, high Reynolds number), the fluid is not diverted, as may be inferred from tests, in some cases till far after the pressure minimum.

determinable according to Theorem 1. It now appears that this velocity w_t for the infinitely small portion of the surface is determined by the flow (fig.10): the quantities A and U in equation 15 would have to be so determined for the time t + dt, that this velocity field merges smoothly into the adjacent region. The writer has not gone into this problem.

Rising off the surface. The writer has been unable to obtain elucidation on what takes place when a body rises off the surface of the liquid (fig.11), when in a region C of a withdrawing body surface, discontinuities of a kind other than those hitherto described appear. The question of such flow phenomena will not be discussed in the present paper.

II Infinitely Flat Bottom.

5. Definitions and Regions.

The fluid completely filling half the space is assumed to possess initially a flat surface. It is assumed to be initially in a state of rest. Motion is set up in the fluid by the movement of the bottom of a body on the surface (three-dimensional problem). We shall consider the limiting case, when the inclination β of the pressure surface relative to the initial surface of the fluid at every point is infinitely small.

The pressure surface, in plan, is assumed as being of finite extent in every direction. The normal velocity V_n at the pressure surface, the order of magnitude of which may be arbitrarily fixed, is assumed to be small as β. The time elapsed since the inception of the process, during which the infinitely small depth of immersion is reached (small, as β), is consequently finite.

The space filled by the fluid is divided into the following regions (fig. 12).

- principal region (principal flow)
- splash root (root flow) [x]
- splash.

The boundary region is common to the principal region and splash root.

The contour of the pressure surface, variable in time is formed (fig. 12).

- at the leading edge, i.e. at positions where it expands (at velocity $\mathcal{W}$,), by the splash foot.
- at positions where it contracts, by the gliding edge.

<u>The equivalent aerofoil motion</u> is regarded as the non-stationary motion of an infinitely thin plate, which at every instant possesses the same form, contour and velocities V_n as the pressure surface, in fluid extending in all directions, at rest at infinity. (fig.13 – right).

---

x For the present limiting case, $B \to 0$, it will be found that the splash root is infinitely thin.

6. Geometrical R[illegible]tion.

In both cases in fig. 13, at the instant under consideration, the same velocity V_n is assumed as being produced by the similar movement of similar plates.

Given similar w_t, the identity of the two velocity fields in the lower half-space at this moment of time follows from the identity and uniqueness of the two boundary conditions.

In the sense of this comparison, the terms "vortex intensity" $2 [v_n \; w_t]$ and "circulation" $\Gamma = 2 \sum$ * will be applied also in the case of the flow in the half-space. It may be readily proved that: If w_t and its directional derivative at the surface is continuous (this being taken as a hypothesis) and if w_t like V_n, is small, as β, then w_n at the free surface is also continuous and small, as β. In general v_n becomes discontinuous (infinitely great) only towards the boundary of the pressure surface.

It is shown further (for the sake of simplicity, for the two-dimensional problem) that the tacit assumption of infinitely small inclination of the free surface is correct. In continuous regions, according to equation (1),

$$\frac{D\alpha}{Dt} = v_n \frac{\partial w}{\partial s},$$

---

* If at infinity the velocity w_∞ is $\neq 0$, then Γ must be put $= 2 \int_A^B (w - w_\infty) \, ds$.

and since $\alpha = 0$, fo[r] $t = 0$ [t]he inclination α of the free surface is small, as $\mathfrak{v}$, hence small, as β. This holds also for the three-dimensional problem.

7. Dynamic Relation.

For the variation Δv_t of the tangential velocity $\mathfrak{v}_t$ during an interval of time Δt, in the continuous regions according to Equation (3), for the two-dimensional problem, the relation :

$$\Delta v_t = \int^{\Delta t} v_n \frac{D\alpha}{Dt} dt = \int^{\Delta\alpha(\Delta t)} v_n d\alpha$$

holds. Consequently Δv_t is small, as $v_n \Delta \alpha$, hence as β^2, so that in the determination of the velocity field (fig. 13) from the boundary conditions, it is negligibly small in relation to the other velocities. Since for the three - dimensional problem, the same result is obtained from equation (6), the following theorem holds:

Theorem 5: In a continuous region of the free surface the tangential velocity $\mathfrak{v}_t$ is invariable in time. Fluid particles, which since $t = 0$ belong to the free surface, have tangential velocity zero. If fluid particles approach the free surface behind a sliding edge with a tangential velocity $\mathfrak{v}_t$ (small, as β), they retain this velocity unchanged.

This conforms exactly with the condition for the continuance of the surface of vorticity behind the equivalent aerofoil. [x]

Production of W_t. According to theorem 5 only fluid particles freshly arriving at the surface can possess velocity W_t. As noted earlier, in general infinitely great velocity v_n would be contingent on arbitrarily given W_t at the sliding edge. According to equation (14) however, v_n must be finite along the sliding edge. Consequently the newly-produced W_t for any moment of time is determined from the condition of continuity at the sliding edge. This, as a result of the identities §. 6 already shown and theorem 5, is identical with the calculation of the surface of vorticity behind the equivalent aerofoil from the condition of continuity at the trailing edge. [xx]

---

[x] c.f. Wagner: "Über die Entstehung des dynamischen Auftriebes von Tragflügeln". Z.a. M.M. 1925. No. 1 par. 2.

[xx]Foot Note: c.f. Wagner. loc. cit. § 1. This comparison is unique, only when the determination of the vortex intensity in the aerofoil problem from the condition of continuity at the edge is unique. This appears unquestionable. Should this however not be the case, e.g. in special cases, the ambiguity would be expressed in an ambiguity in the integral equation formed on the basis of this condition. The integral equation, (e.g. 8), Z.a.M.M. 1925 appears to be unique.

3. Splash Root.

At the leading edge w (in general) becomes discontinuous. A complex co-ordinate z is placed through a point of the leading edge, perpendicular to the latter.

In the case of the equivalent aerofoil the flow function w_s in the region near the leading edge, and the specific, suction force γ exerted here, are given by (C = constant). x

$$w_s = C\sqrt{-z} \qquad (16).$$

and

$$|\gamma| = \frac{\pi}{4}\,\rho\, C^2 \qquad (17).$$

As may be readily shown, γ is small, as β.

In the sliding or impact process, according to equation (1), discontinuous curvature of the free surface is dependent on discontinuous v_n. Since outside the discontinuity $w_t = 0$ at the free surface, let us see whether the velocity field w_1 according to equation (12) (fig.9) obtains in the discontinuous region.

The diameter of the limited region is selected, on the one hand infinitely small (as β) in relation to the dimensions of the pressure surface, so that the flow in the impact or sliding process identical with the aerofoil flow is given by equation (16) but on the other, infinitely

---

xx c.f. Grammel. "Hydrodynamicsche Grundlagen des Fluges". Brunswick. 1917. p. 2.

large (as $1/\beta$) in relation to the thickness δ of the splash, so that $\ln \frac{\delta}{x}$ is negligible in relation to $\sqrt{\frac{x}{\delta}}$ (c.f. equation 13). In this limited region there is smooth transition between the principal flow and the root flow, ($w_1 \equiv w_S$), if along the whole leading edge the thickness of the splash is

$$\delta = \frac{\pi C^2}{16 v_1^2} = \frac{\left|\frac{\Gamma}{2}\right|}{2\rho v_1^2} \qquad (18).$$

[c.f. equation (13) with (16) and (17)].

Since δ is found to be small, as Γ hence, as β^2, the order of magnitude considered is shown subsequently as being permissible.

If, for the principal region towards the leading edge, calculation is made of the form of the free surface according to the flow function w_S in this region, it is seen that in the boundary region also the form of the principal region passes smoothly into that of the splash root equation (8) when (18) is fulfilled.[x] It is seen further, that in the boundary region the inclination of the surface is still infinitely small and that here also W_t is infinitely small compared with v_n, so that here, as in the principal region, the boundary conditions are fulfilled.

9. Summary.

Comparison is made between the impact or sliding process with the motion in the case of the equivalent aerofoil. In the equivalent aerofoil motion, for reasons

---

[x] This condition may also be employed in drawing up

of symmetry, in the principal plane outside the aerofoil the pressure is $p = 0$. From the identity of this condition and the identity of the normal velocity V_n at the pressure surface and the aerofoil and further, the identical condition of continuity at the sliding edge and trailing edge, it follows that in the principal region (lower half-space) in both cases identical flows exist at all times. It is only in the infinitely small region of discontinuity at the leading edge that the splash appears in place of the suction force.

This velocity field composed of two locally different regions, in the limiting case $\beta \to 0$ (provided that equation (13) gives δ continuously variable in time, hence that the conditions of continuity para. 2. are fulfilled), fulfils the geometrical and the dynamical boundary condition along the whole of the free surface (principal region, boundary region, splash root) and fulfils the boundary condition $v_n = V_n$ at the pressure surface. The principal flow and the root flow merge smoothly into each other; the velocity field in the whole region is irrotational and source-free.

The water thrown off in the splash travels at a velocity invariable in time.

Effect of Forces. The pressure on the surface of the bottom at every position in the principal region is exactly of the same order as the pressure on the underside of the aerofoil, hence in this limiting case, half as

great as the difference in pressure between the upper and the under-side.
The root region [up to $\frac{-z}{\delta} \longrightarrow \infty$ in equation (11)] is small, as β. Consequently according to equation (11) the force exerted in the region of the splash root is small, as $\beta^{\frac{3}{2}}$ and, with regard to lift and resistance, may be neglected.

Theorem 6. At all times the magnitude of the lifting force of the surface of the bottom (non-stationary 3-dimensional problem) is half as great as in the case of the motion of the equivalent aerofoil. The position of the forces likewise is in agreement in both cases.

By comparison with the resistance $\mathfrak{W}_t$ of the equivalent aerofoil the suction force is negligible. For the resistance $\mathfrak{W}$ of the sliding surface in the non-stationary three-dimensional problem therefore, the relation

$$\mathfrak{W} = \frac{1}{2} \left(\mathfrak{W}_T - \int S^2 \, du \right) \qquad (19).$$

holds.

The integral is extended over all elements du (fig. 12) of the leading edge.

Energy.

The splash volume $\delta v_1\, dt$ newly-produced over the unit width $\Delta u = 1$ of the splash root in the time dt, has a velocity $2v_1$ and consequently a kinetic energy $dT = \frac{1}{2}\rho . \delta v_1 dt . (2 v_1)^2$. With equation (18) this becomes

$d\,T = \Re_1\,d\,t.\,\frac{\Upsilon}{2}$. This means that the additional work required on the path $v_1\,d\,t$ of the leading edge owing to the absence of the half of the suction force $\frac{\Upsilon}{2}$ compared with the equivalent aerofoil, appears as "nullified" kinetic energy in the splash water.

10. Integral equation of the Impact Motion.

For the limiting case $\beta \rightarrow 0$ of an impact motion let the fluid be assumed as initially at rest with a flat surface. Also (fig.14). let the elevation η_b of all points $\mathfrak{r}_b$ of the bottom surface for the whole period be given, namely $\eta_b = r_b\ (t, \mathfrak{r}_b)$ and thence its normal velocity $V_n = V_n\ (t, \mathfrak{r}_b)$. It is required to determine the variation in time of the contour of the pressure surface; the radius vectors $\mathfrak{r}_\kappa$ of the contour (fig.12) will be given by their value $r_\kappa\ (t_1\ \lambda)$. By solution of the flow problem shown in fig. 12 in the known manner, the velocity $v_n = v_n\ (\mathfrak{r}, t)$ for the free surface is now obtained. This velocity is expressed in the form $v_n\left\{\mathfrak{r}, V_n[t, \mathfrak{r}_b \longrightarrow \longrightarrow r_\kappa\ (t, \lambda)]\right\}$, to indicate that this velocity at position $\mathfrak{r}$ at the time t depends on the given velocities V_n at this time t for all $\mathfrak{r}_b$, so long as $\mathfrak{r}_b$ lies within the momentary contour $r_\kappa(t_1\ \lambda)$ of the pressure surface. The elevation of the fluid particles then is $\eta = \int_0^t v_n\,d\,t$. The fluid particle continues to rise towards the bottom which it finally reaches at the time t_s, when its elevation has become as great as that of this bottom at this position

w_K of the contour of the pressure surface [x] at this time t_o, hence.

$$w_b\ [t_o,\ r_K(t_o,\lambda)] = \int_0^{t_o} v_n \left\{ r_K(t_o,\lambda);\ V_n\ [t,\ w_b - r_K(t,\lambda)] \right\} d\,t \qquad (20).$$

This integral equation, which must be fulfilled for every t_o along the whole contour (hence from $\lambda = 0$ to $2\,\pi$) is used for the determination of $r_K\ (t,\ \lambda)$.

In the sliding motion, according to hypothesis, the path of the sliding edge in time is given. The condition, equation (20) automatically fulfilled at the sliding edge has now to be evaluated for the leading edge. A complication which arises is that v_n depends further on w_t, which in turn must be determined from the condition of continuity at the sliding edge, to be formulated as an integral equation. A closer analysis is omitted here.

The following examples give solutions for this problem in particularly simple cases. The general form of the integral equation will not be retraced.

---

x Closer investigation shows that integration can be made over the root region without notice being taken of the special form of the discontinuity (root flow).

III. EXAMPLES RELATING TO AN INFINITELY FLAT BOTTOM.

11. Uniform Slide - Two-dimensional Problem.

In the case of uniform slide, the lift A is equal to half the lift of the uniformly moving equivalent aerofoil, the resistance W is equal to half the suction force of this aerofoil. Thus for example, in accordance with the aerofoil theory, for a circular pressure surface with height of camber f, depth (profile depth) $2c$ of pressure surface (c.f. fig.15) and angle of incidence β_o of the chord,

$$\left.\begin{aligned} \frac{A}{b} &= \pi\ \rho\ V^2\ (f + c\beta_o) & \frac{W}{b} &= \pi\ \rho\ V^2\ c\beta_o^{\,2} \\ \frac{M}{b} &= \pi\ \rho\ V^2\ c\left(f + \frac{1}{2}\ c\beta_o\right) & \delta &= \frac{\pi}{2}\ c\beta_o \end{aligned}\right\} \qquad (21).$$

M = moment about the leading edge, b = (infinitely large) span. The sliding force is always directed towards the centre of the pressure surface.

For purposes of comparison fig. 15 shows the forces drawn according to scale for four different pressure surfaces. It is seen that for $\beta_o = 0$, slide without resistance is possible. The above equations hold also for negative f, but not for negative β_o, because then the leading edge is awash and another flow appears.

If the sliding edge is imagined as at rest and the fluid in motion, it is seen that (apart from infinitely small quantities of higher order), the free surface is identical with the stream line (path line) which proceeds from the aerofoil in the case of the moving aerofoil.

The flow in the case of the plate, calculated in para. 17 passes into the flow indicated here for $\beta_0 \rightarrow 0$.

18. Uniform Slide. Limiting case of the wide gliding surface.

In the limiting case concerned, the span b [x] (fig. 17) is assumed to be infinitely large in relation to the depth 2c, variable over b, of the pressure surface ($b \gg 2c$). The "centres of gravity" C (fig.16) of the circulation Γ about the individual flat profile elements are assumed as lying, in plan, on a straight line.[x x] The angle of incidence β may be variable over the span.

According to Prandtl's aerofoil theory the relation for Γ is

$$\Gamma = 2\pi V c\,(\beta - \beta_P) \qquad (22).$$

where β_P is the angle of downwash. (Prandtl).

---

[x] All symbols are explained in figs. 16 and 17. In this example z denotes the real co-ordinate. fig. 17.

[x x] When this condition is not met (e.g. in the case of a test) large discrepancies may be expected. On the other hand no serious difficulties would arise if this condition is allowed to lapse.

By varying the height (altitude) of the individual profile elements (the run of η_o) in the case of the sliding surface, the form of the pressure surface would likewise vary, the forward side becoming immersed to a greater or less distance. The present object (in the sense of para. 10) is the relation, equation (27), between the variation in height $\eta_o = \eta_o(z)$ of the profile elements and the variation of the depth $\xi_c = \xi_c(z)$ of the pressure surface, Γ in equation (27) imagined as being replaced by ξ_c according to equation (22) xxx

The velocity v_n of a particle x, z of the surface, owing to the supporting line $\Gamma_\zeta = \Gamma_\zeta(\zeta)$ x and owing to the surface of vorticity $\frac{d\Gamma_\zeta}{d\zeta} d\zeta$ may be calculated directly by Helmholtz' relation (fig. 17). Since the fluid is approaching from infinity at $V = -\frac{dx}{dt}$, the elevation η of a particle is

$$\eta = \int_{t=-\infty}^{t} v_n \, dt = -\frac{1}{V} \int_{x=\infty}^{x} v_n \, dx.$$

Putting in v_n and integrating we get:

$$\eta = \frac{1}{4\pi V} \int_0^b \frac{\Gamma_\zeta \, d\zeta}{\sqrt{x^2 + (\zeta - z)^2}} - \frac{1}{4\pi V} \int_0^b \frac{\sqrt{x^2 + (\zeta - z)^2} - x}{\zeta - z} \frac{d\Gamma_\zeta}{d\zeta} d\zeta \quad (?)$$

(28).

The first term corresponds to the supporting line, the second to the surface of vorticity.

In conformity with the limiting case $b \gg \xi_c$,

---

xxx As approximation βp may be regarded as negligible in relation to β.

let us consider a position for which $b \gg x \gg 2c$ [x]
For small x/b we obtain from equation (23) by development of a series with respect to $\frac{x}{\zeta - z}$ and taking into account the first terms ($\frac{d\,\Gamma\zeta}{d\zeta}$ assumed continuous). [x x]

$$\eta = \frac{\Gamma}{4\pi V}\left[\int \frac{\left(\frac{\Gamma\zeta}{\Gamma} - 1\right) d\zeta}{\left|\zeta - z\right|} + \ln \frac{4z\,(b - z)}{x^2}\right] - \frac{\Gamma}{2\pi V} + x\beta P \qquad (24).$$

Γ is the circulation at the position $\zeta = z$ (cf. fig. 17).

Compared with the flow past the plate of finite width, the flow in the region of the profile cross-section is inclined at an angle βP. (fig. 16). Otherwise however, as may be shown, in the immediate vicinity of the profile ($b \gg x$) the form of the surface is approximately the same as in the two-dimensional problem. Consequently, (fig.16).

---

x By thus putting 2c doubly small compared with b, which is necessary because of equations (24) and (26), it may be anticipated that, with finite spans larger discrepancies will arise than in the case of Prandtl's aerofoil theory. On the other hand the assumption $b \gg x$ does not appear to be too restrictive, since even for $x \geq + c$ or for $x \leq - 1.8\ c$ the elevations y of the free surface for infinitely wide aerofoils are hardly different from those given by equation (26).

x x For $x = 0$ the second integral of equation (23) gives $\eta = \frac{\Gamma}{2\pi V}$. The term $x\,\beta P$ may be added without mathematical proof, since $\beta\Gamma$ beyond $x = 0$ is continuous.

$$r_o = \eta + yb = \infty - x\,\beta\,P \qquad (25).$$

$y_b = \infty$ may be determined for the limiting case $x \gg 2c$ and for $\beta \to 0$ from equation (80).

$$yb = \infty = \frac{\Gamma}{4\pi V}\left(1 + \ln\frac{4x^2}{c^2}\right) \qquad (26).$$

This and η according to equation (24) are put into equation (25): ※ ※ ※

$$\eta_o = \frac{\Gamma}{4\pi V}\left[\int_0^b \frac{\left(\frac{\Gamma\zeta}{\Gamma} - 1\right) d\,\zeta}{|\zeta - z|} + \ln\frac{16z\,(b - z)}{c^2} - 1\right] \qquad (27).$$

There is still the resistance to be considered. Using $\frac{d\,A}{d\,z}$ to denote the lift "density", then, since the force is inclined at β at every position, the resistance is

$$W = \int_0^b \beta\,\frac{d\,A}{d\,z}\,dz = \int_0^b \beta_P\,\frac{d\,A}{d\,z}\,dz + \int_0^b (\beta - \beta P)\,\frac{d\,A}{d\,z}\,dz \qquad (28).$$

The first component, as in the case of the aerofoil, corresponds to the kinetic energy of the water moving "downwards" behind the sliding surface. The second is used for the production of the splash water. This component can be avoided by cambering the profile.

※ ※ ※ The last term in the brackets, viz -1, in accordance with our limiting case is small compared with the first two terms. It seems right, however, to take this term into account in a numerical calculation.

13. <u>Uniform Slide.</u> <u>Limiting case of the long sliding surface.</u>

Let us consider (fig.18) the limiting case $\frac{l}{2c} \rightarrow \infty$. If the bottom (infinitely flat) is keeled and if the water in the forward portion of the sliding surface requires a length l_s in order to reach the lateral edge of the bottom, then $\frac{l_s}{2c}$ will also $\rightarrow \infty$.

If an infinitely long plate of width $2c$ moves in a fluid extending in all directions perpendicularly to its plane at a velocity V_n, the behaviour of the surrounding water may be reduced to that of a mass of water carried along, of the quantity $\pi\ \rho\ c^2$ (per unit length of the plate). Behind the sliding surface in the lower half-space, there is produced every second on an element of length V_n the velocity field about a plate of width $2\ c_0$ of the step moving downwards at the velocity $V_n = V\ \beta_0$. The momentum transmitted to the water per second:

$$B/sek = V \cdot \frac{1}{2} \pi\ \rho\ c_0^2 \cdot V\ \beta_0 = P \qquad (59).$$

produces thd sliding force P of the same value.

In the case of the infinitely flat keeled bottom, over the element l_s, the pressure distribution with $V_n = V\beta$ and $d\ V_n/d\ t = - V^2 \frac{d\beta}{dz}$ is given by equation (45); behind l_s pressures only appear with variable inclination of the bottom: $p = - \rho\ V^2 \frac{d\beta}{dz} \sqrt{c^2 - x^2}$.

For the production of the energy remaining in the water behind the bottom per metre of path, the work

$$A/\text{Meter} = \frac{1}{2} \cdot \frac{1}{2} \pi \rho c_0^2 \cdot (V \beta_0)^2 = W_0 \qquad (39).$$

is consumed, which corresponds to a resistance W_0 of the same value. In the case of the bottom with straight keel $\beta = \beta_0$, the value of the total resistance $W = P\beta_0$ (cf. equation 2a) is just twice that of W_0. A quantity of energy of the same value is thus absorbed in the splash water, which is produced in the region l_1. If the angle β in the region l_1 is reduced, this loss of energy is reduced in the same measure and eventually avoided altogether. The distribution of the forces and the resultant P are shown for 2 cases in fig. 19.

14. Impact of keeled surfaces.

Let the initially flat water surface be considered as moving upwards towards the bottom, imagined as at rest (fig.20). Let the velocity V of the water at infinity be given as a function of the time : $V = V(t)$.

In this impact process, the velocity at the free surface in the principal region is directed vertically ($v_t = 0$). The velocity at the bottom drops in the direction of the bottom. The flow (fig. 20) uniquely given by these boundary conditions is in (infinitely) close agreement with the flow of a fluid extending to infinity about a flat plate at rest (cf. para. 6.). The width 2c of the plate is the momentary width of the pressure surface. The velocity of the water at a position $x > c$ of the surface is:

$$v_n = \frac{V}{\sqrt{1 - \frac{c^2}{x^2}}} \tag{31}$$

The elevation η of the water, calculated from the moment of immersion ($t = 0$) is

$$\eta = \int_0^t v_n \, d\,t = \int_0^t \frac{V \, d\,t}{\sqrt{1 - \frac{c^2}{x^2}}} \tag{32}$$

The width $2\,c$ of the pressure surface increases with time: $c = c\,(t)$. Selecting for the moment c as independent variable, hence $t = t\,(c)$ and also $V = V\,(c)$, we may put $d\,t = \frac{d\,t}{d\,c} \cdot d\,c$, thus (owing to u, cf. below):

$$\eta = \int_{c=0}^{c \leq x} \frac{V \frac{d\,t}{d\,c}}{\sqrt{1 - \frac{c^2}{x^2}}} \, d\,c = \int_{c=0}^{c \leq x} \frac{u\,(c)\, d\,c}{\sqrt{1 - \frac{c^2}{x^2}}} \tag{33}$$

At the moment, when the water particle at position x reaches the contour of the pressure surface, c has become $= x$ and $\eta = \eta b$:

$$\eta b = \int_0^x \frac{V \frac{d\,t}{d\,c}}{\sqrt{1 - \frac{c^2}{x^2}}} \, d\,d = \int_0^x \frac{u\,(c)\, d\,c}{\sqrt{1 - \frac{c^2}{x^2}}} \tag{34}$$

The two dependent variables V and $\frac{d\,t}{d\,c}$ have been combined in

$$u = u\ (c) = \frac{V}{\frac{dc}{dt}} = \frac{V}{v_z} \qquad (35).$$

Equation (34) must hold for all x and is to be taken as integral equation for the determination of u (c), (cf. para. 10). The infinitely small u is a purely geometrical quantity; it depends only on ηb (x) but not on V = V (t). It will always be possible to represent the given form of bottom by the series.

$$\eta b = \beta\ x + \beta_1\ x^2 + \beta_2\ x^3 + \beta_3\ x^4 + \beta_4\ c^5 + \ldots. \qquad (36a).$$

The solution of equation (34) is then written in the form (trial by substitution !)

$$u = u(c) = \frac{2}{\pi}\,\beta\beta + \beta_1\ c + \frac{4}{\pi}\,\beta_2\ c^2 + \frac{3}{2}\,\beta_3\ c^3 + \frac{16}{3\pi}\,\beta_4\ c^4 + \ldots \qquad (36b).$$

If V_n is now given, for example, as function of t, c = c (t) may be determined by equation (35), namely from

$$\int_0^c u\ (c)\ d\ c = \int_0^t V_n\ (t)\ d\ t \qquad (37).$$

The form of the water surface $\eta = \eta\ (x)$ may be determined from equation (33), when, after integration, c is regarded as given and x as variable. For example, for a straight keeled bottom we get.

$$\eta = \frac{2}{\pi} \beta . x \text{ arc sin } \frac{c}{x} \qquad (38).$$

To determine the force P on the body, the fluid is assumed as being initially at rest and the body moving relatively to the fluid at the velocity $V = V\ (t)$.

The "momentum" of the fluid is

$$B = \frac{\pi}{2} \rho\ c^2\ V \qquad (39).$$

By differentiation with respect to t, using equation (35) we get

$$P = \pi\ \rho\ c\ \frac{V^2}{\upsilon} + \frac{\pi}{2}\ \rho\ c^2\ \frac{dV}{dt} \qquad (40).$$

For the special case when a body of given mass m with initial velocity V_o strikes the water, the momentum of the fluid (equation 39) is put as equal to the momentum $m\ (V_o - V)$ given up by the body, and we get

$$V = \frac{V_o}{1 + \mu} \qquad (41).$$

where $\mu = \frac{\pi\ \rho\ c^2}{2\ m}$ (42).

Putting $P = - m \dfrac{d\,V}{d\,t}$ into equation (49), and using equations (41) and (42) we obtain

$$P = \frac{\pi\,\rho\,c\,V_o^2}{(1+\mu)^3\,.\,u} \qquad (43).$$

For the straight-keeled bottom $\eta b = B_x$, hence $u = \dfrac{2}{\pi}\,\beta$, for example, this gives

$$P = \frac{\pi^2}{2}\;\frac{\rho\,c\,V_o^2}{(1+\mu)^3\,.\,\beta} \qquad (43a).$$

It may be of interest to determine the form which should be given to the bottom in the case of a body of given mass, so that $P = P^o$ shall be constant in time. In this case we must put $u = u\,(c)$ in accordance with equation (43) into equation (34) and integrate for constant $P = P_o$, but it must be borne in mind that μ according to equation (42) depends on c. The integration gives

$$\eta b = \frac{m\,V_o^2}{2\,P_o}\;\frac{\mu_x}{1+\mu_x}\left[1 + \frac{3}{2(1+\mu_x)} \pm \frac{3\ln(\mu_x + \sqrt{1+\mu_x})}{2\sqrt{\mu_x}\,(1+\mu_x)^3}\right], \qquad (44).$$

where $\eta_x = \dfrac{\pi\,\rho\,x^2}{2m}$. A bottom of this kind is represented in fig. 21.

We now calculate the pressure distribution at the bottom. For $|x| < c$ the velocity potential is $\varphi = - V\sqrt{c^2 - x^2}$. The fluid pressure p is *

$$p = - \rho\left[\frac{\partial\varphi}{\partial t} + \frac{1}{2}\,v_x^2 + F\,(t)\right]. \qquad (45).$$

---

* Lamb. loc. cit. p. 20.

with $\varphi = \varphi\,(V,\ c)$ and $v_x = \frac{\partial \varphi}{\partial x}$, we obtain with equation (35):

$$\frac{p}{\rho} = \frac{V^2}{u}\,\frac{1}{\sqrt{1 - \frac{x^2}{c^2}}} + \frac{d\,V}{d\,t}\sqrt{c^2 - x^2} - \frac{1}{2}\,\frac{V^2}{\frac{c^2}{x^2} - 1} \qquad (45).$$

The last term, due to the square of the velocity, is infinitely small compared with the first two terms which are due to $\frac{2\,\varphi}{\partial\,t}$ (this does not hold for the region of the splash root).

For the case of the impact of a body of given mass m, we obtain from equations (41) and (43):

$$p = \frac{P}{\pi c}\left[\frac{1 + \mu}{\sqrt{1 \pi \frac{x^2}{c^2}}} - 2\mu\sqrt{1 - \frac{x^2}{c^2}} - \frac{u(1 + \mu)}{2\left(\frac{c^2}{x^2} - 1\right)}\right] \qquad (46).$$

The first term due to $d\,c/d\,t$, gives large positive pressures at the edge of the pressure surface, whereas the second term gives negative pressures with elliptical distribution produced as a result of the fact that simultaneously with the body, the water is retarded in the course of the impact. Since the pressures at the centre of the body corresponding to the first term are relatively small, it is highly probable that, in cases occurring in practice there will be negative pressure at the centre of the bottom, whilst at the edge of the pressure surface very large positive pressures which, taken in their entirety predominate, cause retardation of the impact. When the pressure surface finally reaches the lateral edge of the bottom, the impact process is ended.

We now calculate the thickness δ of the splash. In the case of the flow for the plate (fig. 20) the origin of the complex co-ordinate z is placed (cf. para. 8) at the point $x = c$. The complex flow function of the flow for the plate $w_p = i V \sqrt{2 c z} \sqrt{\frac{z}{2c} - 1}$ for small z becomes $w_p = V \sqrt{2c} \sqrt{-z}$. By comparison with equation (13) we obtain

$$\delta = \frac{\pi}{8} c u^2 \tag{47}$$

The same value δ is obtained, if in the range of small z the form of the surface of the splash root is taken as equal to that of the surface of the principal flow or if the energy expended by the body on impact is put equal to the kinetic energy of the principal flow plus that of the splash flow (cf. para. 8). It should be noted that at the beginning of the impact, half the kinetic energy of the fluid is contained in the splash; towards the end of the impact the splash contains nearly the whole of the kinetic energy given up by the body.

In order to show the transition from plate flow to splash flow with _finite_ keel angle, the form of the free surface (equation 33) and the pressure distribution (equation 45) for the root region, corresponding to the plate flow for finite angle $\beta = 0.1$ are compared in fig. 22 with these factors for the splash flow. (equations 9 and 10).

Fig. 23 shows the pressure distribution during the impact of a body of $2b = 2$ m width, 1100 kg weight per metre length with a velocity of drop $V_0 = 5$ m/sec. The figures, give the impact forces in tons.

15. Touching of a step. Two dimensional problem.

As an example of a non-uniform sliding motion, let us consider the touching of a step (fig. 24). The factors assumed as given are: the velocity W *, constant in time, of the step and the normal velocity $W_u = W(\beta + \kappa)$; the depth 2c (variable in time) of the pressure surface compared with the path Wt travelled by the step in the time t; the time t in which C = 1 is selected for the sake of shortness. The purpose of the calculation, namely, (in the sense of para. 10) the determination of β and κ, is achieved by determining the velocity field and the calculation of the free surface (in particular of r_a).

It is seen that a fluid motion with centre of similitude (cf. para 16) is produced, the latter being the point of the original surface at which the step touched

* In this example the velocity of the body is indicated by W instead of by V as hitherto.

(in fig. 24 the starting point of the vector $W\,t$). On the portion of the free surface swept by the sliding edge $x_0 \geqq x \geqq c$, a horizontal velocity v_t (c.f. para. 7) is produced, which, in the case of the equivalent aerofoil motion corresponds to a surface of vorticity with velocity jump $u = 2\,v_t$. Since the value of u at every position is invariable in time (Theorem 5), but owing to the similitude, the circulation about the surface of vorticity and the length of the latter increase with time, the value $u = 2\,v_t$ of the discontinuity on the portion $x_0 \geqq x \geqq c$ must also be constant in place.

In order to determine the velocity field it is necessary first to know the relation between W_n and u. *
In fig. 25 the co-ordinates of the individual vortex filaments are denoted by ξ and the circulation about a filament by $u\,d\,\xi$. In the conformal transformation of the plate flow about a cylinder all quantities are denoted by the capital letters. Namely. **

$$X = x + \sqrt{x^2 - 1} \quad (43). \text{ and } \Xi = \xi + \sqrt{\xi^2 - 1} \quad (49).$$

The velocity field is divided into the two fields I and II.

---

* From the author's paper, loc. cit, equation (8), equation (54) is obtained directly with the notation for u = const. The process has been repeated here in such a ma[illegible]r that the velocities v_n at the free surface are also obtained (equation 55).

** c.f. Wagner, loc. cit. equation (4).

Velocity field I due to circulation about the surface of vorticity the aerofoil imagined as at rest with fluid at rest at infinity. The velocity at a position X of the surface in the system of the circular contour due to a vortex filament with $\Xi = \frac{1}{\bar{\Xi}}$ ※※※ is

$$d\,V_{n\,I} = -\frac{d\,\Gamma}{2\,\pi\,(\Xi - X)} = \frac{d\,\Gamma}{2\,\pi\,(X - \Xi_1)}\,.$$

Since the circulation remains unchanged in the conformal transformation $d\,\Gamma = U\,d\,\Xi = u\,d\,z$

Consequently

$$V_{n\,I} = -\frac{u}{2\pi}\int_1^{X_0}\left(\frac{1}{\Xi - X} + \frac{\Xi}{\Xi X - 1}\,\frac{d\,\xi}{d\,\Xi}\right)d\,\Xi \qquad (50).$$

We introduce $\frac{d\,\xi}{d\,\Xi}$ according to equation (49) and integrate. With conformal transformation $v = V\,\frac{d\,X}{d\,x}$. Equation (50) is therefore multiplied by $d\,\frac{d\,X}{d\,x}$ according to equation (45) and we obtain:

$$v_{nI} = -\frac{u}{2\pi}\left(\frac{X^2 + 1}{X^2 - 1}\ln X_0 + X\,\frac{X_0 - X_0^{-1}}{X^2 - 1} + \ln\left|\frac{X_0 - X}{X_0\,X - 1}\right|\right) \quad (51).$$

At the sliding edge, for $X \longrightarrow 1$:

$$v_{n\,I} = \lim_{x \to 1}\left[-\frac{u}{2\pi}\,\frac{1}{X^2 - 1}\,\left(X_0 - X_0^{-1} + 2\ln X_0\right)\right] \qquad (52).$$

※※※ c.f. Wagner. loc. cit. equation (5).

Velocity field II. the circulation -free field about the aerofoil moving at W_n with fluid at rest at infinity. Here * $v_{n\,II} = W_n \left(\frac{x}{\sqrt{x^2-1}} - 1\right)$ *. In place of x, the variable X according to equation (49) is introduced:

$$v_{n\,II} = W_n \frac{2}{X^2 - 1} \tag{53}$$

Total field. In order that, in accordance with condition para. 7, at the sliding edge (for $X \to 1$) $v_n = v_{nI} + v_{nII}$ shall remain constant, according to equations (52) and (53) the relation

$$u = 2\,v_t = \frac{4\,\pi\,W_n}{X_o - X_o^{-1} + 2.\ln X_o} \tag{54}$$

must hold. We introduce W_n, according to this equation into (53) and with (51) form $v_n = v_{nI} + v_{nII}$

$$v_n = -\frac{u}{2\pi}\left(\frac{X_o - X_o^{-1}}{X + 1} + \ln\left|\frac{X - X_o}{X - X_o^{-1}}\right|\right) \tag{55}$$

The elevations η are most easily calculated by equation (75), where in the limiting case $s = x_o + x$. Thus, for example, in the region of negative x:

$$\eta = (x_o - x) \int_{-\infty}^{x} \frac{v_n\,t}{(x_o - x)^2}\,d\,x \tag{56}$$

---

* Lamb. loc.cit. p. 32.

Thus w_2 for $x = -1$ is also known. Since $2 c \beta - W t \kappa = n_1$ (fig. 24) and $W_n = W(\kappa - \beta)$ the angles k and β are now individually known.

In order to determine the thickness of the splash δ, the origin of the co-ordinates is placed in the splash root ($x = -1$) and v_n (equation 55) developed with respect to powers of this new abscissa. This velocity v_n is compared with $v_n = \left| \frac{d\, w_2}{dz} \right|$ for real $z > 0$ according to equation (13), when (fig. 24) for the velocity v_1 of the splash root $v_1 = \frac{x_o + 1}{x_o - 1} W$. Using the earlier notation we obtain.

$$\delta = \frac{\pi}{2} \left(\frac{X_o - 1}{X_o + 1} \right)^4 \left(\frac{\kappa + \beta}{1 + \frac{2 \ln X_o}{X_o - X_o^{-1}}} \right) \qquad (57).$$

We determine the force (Σ P) in the equivalent aerofoil. ※※
We have ※※※

$$\int_{Fv + Fa'} \varphi \, d F = \pi W_n + \int_1^{x_o} \left(\xi - \frac{1}{\Xi} \right) u \, d \xi = \frac{B}{\rho} \qquad (58).$$

where $b = 1$, $v_p \sin \beta = W_n$, $\Gamma = u\, d\xi$ and instead of x_o, X_o, the symbols ξ, Ξ have been introduced. In addition the integral over all vortex filaments $u\, d\xi$ has been indicated. B is the "momentum" of the fluid.

※※ Wagner. loc. cit. equation (24).

※※※ Wagner loc.cit. equations (28) (29) (31).

$(2\,P) = \frac{D\,B}{D\,t}$. Since (c.f. para. 16) both the potential φ and the surfaces increase linearly with time, B is proportional to t^2, hence $\frac{D\,B}{D\,t} = \frac{2\,B}{t}$. According to fig. 24 we may put

$$t = \frac{x_0 - 1}{W}.$$

By integration of equation (58) we finally obtain the sliding force.

$$P = \frac{\rho\,W\,u}{8\,(x_0 - 1)}\,(1 + X_0)^2\,(1 - X_0^{1-x}) \qquad (59).$$

The author has made the calculation for the example $x_0 = 2{\cdot}5$ c. The result obtained was : (equation 54) $v_t = 0 \cdot 813\,W_n$, further after calculating η by equation (56), $\beta = 1 \cdot 25\,x$; the surface for $\beta = 0 \cdot 21$ is shown in fig. 26. In the region about $x = x_0$ owing to discontinuity (inclination of the surface not infinitely small) the calculation does not hold. In this region however, theorem 3 and equations (64) and (65) even without calculation, provide a good basis for making the drawing of the surface. The velocity W is indicated at various positions of the surface.

IV. VARIOUS LIMITING CASES.

16. Fluid motion with centre of similitude.

The assumption of infinitely small inclination of the surface is dropped in this case. If, for example, the straight-keeled bottom immerges into the fluid at a velocity V_0, constant in time, this fluid motion, as may be readily seen, is subjected to the general law of similitude: The forms of the surface at the different times are geometrically similar and the same velocities then occur at positions corresponding to each other.

If the fluid is imagined as moving towards the bottom at rest, the lowest point of the bottom will represent the centre of similitude O.

In fig. 27 the different fluid particles of the free surface are denoted by their abscissas ξ at the time $t = 0$. Their position at a subsequent time t is given by the radius vector $\mathfrak{r} = \mathfrak{r}(\xi, t)$ relative to O. The velocity is $\mathfrak{w} = \mathfrak{w}(\xi, t) = \frac{\partial \mathfrak{r}}{\partial t}$. The interpretation of fig. 27 according to the law of similitude is:

If at the time t, the fluid particle ξ is in the position $\mathfrak{r}$ and has the velocity $\mathfrak{w}$, then at the time n t (with $n < 1$) the fluid particle n ξ was in position n $\mathfrak{r}$ and had the velocity $\mathfrak{w}$. Expressed as a mathematical formula:

$$\frac{\mathfrak{r}}{t} = \frac{\mathfrak{r}}{t}\left(\frac{\xi}{t}\right) \quad \text{and} \quad \frac{\partial \mathfrak{r}}{\partial t} = \mathfrak{w} = \mathfrak{w}\left(\frac{\xi}{t}\right)$$

By partial differentiation of these two equations with respect to t

$$\frac{\mathfrak{w}}{t} - \frac{\mathfrak{r}}{t^2} = -\frac{d\frac{\mathfrak{r}}{t}}{d\frac{\xi}{t}}\frac{\xi}{t^2} \qquad (60): \quad \text{and} \quad \frac{\partial \mathfrak{w}}{\partial t} = -\frac{d\mathfrak{w}}{d\frac{\xi}{t}}\cdot\frac{\xi}{t^2} \qquad (61)$$

For a definite moment of time t, we introduce instead of ξ as independent variable the length of arc $s = s(\xi)$ of the free surface: $\mathfrak{w} = \mathfrak{w}(s)$, $\mathfrak{v} = \mathfrak{v}(s)$. Equations (60), (61) now acquire the form

$$\mathfrak{v} t - \mathfrak{w} = -\frac{d\mathfrak{w}}{ds} \cdot \xi \frac{ds}{d\xi} \quad (62) \quad \text{and} \quad \frac{\partial \mathfrak{v}}{\partial t} = -\frac{d\mathfrak{v}}{ds} \cdot \frac{ds}{d\xi} \cdot \frac{\xi}{t}. \quad (63).$$

Equation (63) shows that $\frac{d\mathfrak{v}}{ds}$ is in the direction $\frac{\partial \mathfrak{v}}{\partial t}$, hence, owing to the dynamic boundary condition, perpendicular to the surface $\frac{d\mathfrak{w}}{ds}$. By differentiation of equation (62) with respect to s

$$t\frac{d\mathfrak{v}}{ds} - \frac{d\mathfrak{w}}{ds} = -\frac{d^2\mathfrak{w}}{ds^2}\,\xi\,\frac{ds}{d\xi} - \frac{d\mathfrak{w}}{ds}\,\frac{d}{ds}\left(\xi\,\frac{ds}{d\xi}\right).$$

The first terms on both sides of the equation are perpendicular to $\frac{d\mathfrak{w}}{ds}$. Consequently $\frac{d}{ds}\left(\xi\,\frac{ds}{d\xi}\right) = 1$.

With constants A and B, $\xi\,\frac{ds}{d\xi} = s + A$ and thence $s + A = B\,\xi$.

With $\frac{d\mathfrak{w}}{ds} = \mathfrak{w}_t$, equation (62) becomes $\mathfrak{v}t = \mathfrak{w} - \mathfrak{w}_t\,(s + A)$.

In the present case (fig. 27) $\xi = 0$ for $S = 0$ and $\frac{ds}{d\xi} = 1$ for $\xi = \infty$. Consequently

$A = 0$, $B = 1$, hence.

$$\mathfrak{v}\,t = \mathfrak{w} - \mathfrak{w}_t\,s \quad (64) \quad \text{and} \quad s = \xi \quad (65).$$

The interpretation of equation (64) according to fig. 28) is: With known form of the free surface, for a point in the latter the other end point of the vector $\mathfrak{w}t$ is obtained by drawing from the centre of similitude the length of arc s of the free surface, reckoned from the end of the splash, parallel to the tangent to the surface.. ※ Equation (64), (by multiplication by $\mathfrak{n}_n$ and $\mathfrak{n}_t$) may be replaced by the two scalar relations:

$$v_n t = \mathfrak{n}_n \mathfrak{r} \qquad (66) \qquad \text{and} \qquad v_t t = \mathfrak{n}_t \mathfrak{r} - s \qquad (67).$$

※ Foot Note: It should be mentioned, that the boundary of the region of the auxiliary quantity $h = \int^{z} \sqrt{\frac{dv}{dz}}\, dz$ corresponding to the fluid space is rectilinear. The variation of the velocity in the direction of the free surface, according to equation (63) is perpendicular to the direction of the surface, so that with $v = v_x - i\, v_z$ the product $d\,v\,d\,z$ at the free surface is in the direction of the imaginary axis. Hence along the free surface the direction of $\sqrt{d\,v\,d\,z} = \sqrt{\frac{dv}{dz}}\, d\,z$ is $\pm \frac{\pi}{4}$. Along the bottom it is in the direction of the real axis (fig.34). In deriving a second complex relation independent of the one above, there is a temptation to make use of the fact that (c.f. fig. 28) $\frac{d\,z}{z - v_x - iv_y}$ at the free surface is real. But since this contains the quantity conjugate with v, the author has been unable to find a conclusive solution for the whole problem.

Thence with $d\psi = v_n\, d\, s$ and $d\,\varphi = v_t\, d\, s$,

$\psi\, t = 2\, F +$ konst (68) and $\varphi\, t = \frac{1}{2}\,(s^2 - r^2) +$ konst

(69).

ψ = flow function, φ = potential. F is the shaded area in fig. 26.

For subsequent purposes a further relation is derived. The fluid particle ξ_0 which, at the moment t_0 was situated at the position w_0 of the surface, has in the course of t_0 become displaced by $w_0 - v_x\, \xi_0 = \int_0^{t_0} w\, d\, t$. According to the law of similitude (fig.27) the velocity w of this particle at a smaller time t was of the same value as the momentary velocity of the particle $\xi = \frac{t_0}{t}\, \xi_0$. We may therefore introduce $d\, t = -\frac{1}{\xi^2}\, t_0\, \xi_0\, d\xi$ into the integral.

Owing to equation (65) $s = \xi$, therefore

$$w_0 - v_x\, s_0 = s_0\, t_0 \int_{s_0}^{\infty} \frac{w}{s^2}\, d\, s \qquad (70).$$

Multiplying this equation by v_η (fig.23) we obtain equation (75). The suffix 0 being no longer necessary, is omitted.

Impact force.

Let $\mathfrak{P}_t$ denote the impact force on the body, F_T the wetted area of the bottom, F_0 the free surface of the fluid. If the element of area d f of the fluid is imagined as directed outwards, $\mathfrak{P}_t$ becomes: *

$$\mathfrak{P}_T = - \rho \frac{D}{Dt} \int\limits_{F_T + F_0} \varphi \, d f .$$

Since, owing to similitude, φ and F increase linearly with the time t, the value of the integral is proportional to t^2. Its variation in time is consequently the $\frac{2}{t}$ th. of its value at the time being.

Hence:

$$t \, \mathfrak{P}_T = - 2 \rho \int\limits_{F_T} \varphi \, d f - 2 \rho \int\limits_{F_0} \varphi \, d f \qquad (71).$$

Owing to the linear increase of $\mathfrak{P}_t$ and to the constant impact velocity V_0 of the body, the energy E imparted to the body is $E = - \frac{1}{2} \mathfrak{w}_0 \cdot t \cdot \mathfrak{P}_t$. Consequently **

$$- \mathfrak{w}_0 \, t \, \mathfrak{P}_T = \rho \int\limits_{F_T} \varphi \, \mathfrak{v} \, d f + \rho \int\limits_{F_0} \varphi \, \mathfrak{v} \, d f + \rho \int\limits_{F_\infty} \varphi \, \mathfrak{v} \, d f \qquad (72).$$

As may be seen, the integral over the infinitely distant boundary surface F_∞ becomes infinitely small, and may be omitted. At the surface of the body F_T, owing to the geometrical boundary condition, $\mathfrak{v} \, d f = \mathfrak{w}_0 \, d f$. Therefore

---

* Wagner. loc. cit. para. 6.

** Lamb. loc. cit. p. 51.

we remove $\mathfrak{w}_o$ from the first integral on the right hand side, multiply equation (71) by $\mathfrak{w}_o$ and remove the first integral on the right-hand side from equations (71) and (72):

$$\mathfrak{w}_o \, \mathfrak{P}_t \, t = \rho \int_{F_o} \varphi \, (\mathfrak{v} + \mathfrak{w}_o) \, d f \qquad (73).$$

Since, by equations (64) and (69) $\mathfrak{v}$ and φ are known as functions of the form of the free surface, the component of the impact force, $\mathfrak{P}_t$ in the $\mathfrak{w}_o$ direction is determined by the form of the surface.

Iteration. By means of equation (64), the form of the free surface in the case of impact, may be determined by improving an estimated form by a process of iteration. For example: An estimate is made of the form of the free surface $\eta = \eta(x)$ and the velocity v_x at the free surface determined by the equation

$$t \, v_x = x - s \frac{d x}{d s} \qquad (74).$$

obtained from equation (64) by multiplying by $\mathfrak{u}_x$. From this and from the boundary condition at the bottom, using Green's theorem ✻, the velocity v_y corresponding to the source-free and irrotational flow at all positions

---

✻ A similar calculation is made in Z a M M 1925. No. 2? g...k. "Der Überfall über ein Wehr". ("Fall over a weir").

of the free surface is determined and this velocity v_y used for the calculation of new η :

$$\eta = s\,t \int_s^\infty \frac{v_y}{s^2}\, d\,s \tag{75}$$

(cf. equation 70). With this new form of the surface, the process may be repeated: new v_x by equation (74), new v_y by Green's theorem, new η by equation (75).

This method has been employed by the author (after considerable modification in order to meet the special conditions of the example under consideration) for the calculation of the surface on the impact of a symmetrical bottom with keel angle $\beta = 18^\circ$. (fig. 29). From this and with equation (73) the impact force is found to be :

$$P = 49{,}8\ T\ \rho\ V_0^2 \tag{76}$$

Contrary to para. 14, the impact force P refers to the depth of immersion T (fig. 29) since the width of the immersion surface with finite keel angle is an undefined quantity.

Let us estimate the relation of the impact force P for a constant velocity of immersion V_0 for the whole range of keel angles from $\beta = 0$ to $\frac{\pi}{2}$. For this we generalize equation (48a) ($\mu = 1$, $\frac{2c}{\pi}\beta = T$) by introducing a value K:

$$P = K\,\frac{\pi^2}{4}\,\rho\,V_0^2\,\frac{T}{\beta^2} \tag{77}$$

For $\beta \to 0$, equation (43a) holds, hence $K = 1$.

For $\beta = 28^\circ$, equation (76) gives $K = 0.625$.

Finally the curve for the value K according to equation (83) for the immersion of a knife-edge is given in fig. 30.

As an approximation, use might be made of the parabola $K = \left(1 - \frac{2\beta}{\pi}\right)^2$, hence with equation (77),

$$P = \tfrac{1}{4}\,\pi \left(\frac{\pi}{2\beta} - 1\right)^{-2} \rho\, V_0^2\, T \qquad (78).$$

17. Uniform Slide. Two-dimensional Problem.

For the two-dimensional problem of the uniform slide of a flat plate (fig. 31), according to the Schwarz-Christoffel method.* the relations

$$\frac{dz}{dt} = A\,\frac{t - a}{t - 1} \qquad \frac{d}{dt}\left(\ln - \frac{dz}{dw}\right) = \frac{i}{\sqrt{t^2 - 1}\,(t - a)}$$

hold, consequently:

$$\frac{dz}{dt} = \frac{dz}{dw}\,\frac{dw}{dt} = \frac{at - 1 \pm i\sqrt{1 - a^2}\sqrt{t^2 - 1}}{t - a} \cdot A\,\frac{t - a}{t - 1} \qquad (79).$$

$$\frac{z}{A} = a\,(1 + t) - (1 - a)\ln\frac{1 - t}{2} \mp \sqrt{1 - a^2}\sqrt{1 - t^2} \mp$$
$$i\sqrt{1 - a^2}\,\ln\left(- t - \sqrt{t^2 - 1}\right) \qquad (80).$$

---

* A mathematically similar example is worked out in "Besant and Ramsay. Hydrodynamics Vol. II. p. 142.

At the contour the top, middle and bottom signs hold along $\widehat{12}$, $\overline{24}$, $\widehat{45}$ respectively. For logarithms the cardinal value must be introduced. It should be observed along $\widehat{45}$, that $\widehat{45}$ bounds a region with positive imaginary part. $a = \cos\beta$. The constant A may be represented, for example, by the thickness δ of the splash: $\delta = \pi\,(1 - a)\,A$. The sliding plate lies (logarithmically) infinitely high above the level of the fluid at infinity.

From the readily calculated pressure distribution, by integration, the sliding force P (V = velocity of slide) is obtained:

$$P = A\,\pi\,\rho\,V^2 \sin\beta = \rho\,\delta\,V^2 \cot\frac{\beta}{2} \qquad (81).$$

In the limiting case $\beta \to 0$ agreement in every respect with Section II is obtained.

18. Immersion of a knife-edge.

Let a plane velocity field, fig. 32, have $v_t = 0$ (hence also $\varphi = 0$) along A C and $A_1\,C_1$ and $v_n = V\cos\beta = V\cos\beta$ = const along CDC_1. Its calculation presents no difficulty. In particular, the "momentum" B is

$$B = \rho\left|\int_F \varphi\,d\,f\right| = \rho\int_{-C}^{+C}\varphi\,d\,x = \rho\,V\,Y^2\cot\beta\ \chi,$$

$$\chi = \left(\frac{\pi - 2\beta}{\sin 2\beta}\,\frac{\pi}{\left[\Gamma\left(1-\frac{\beta}{\pi}\right)\cdot\Gamma\left(\frac{1}{2}+\frac{\beta}{\pi}\right)\right]^2} - 1\right) \qquad (82).$$

The vector d f is an element of the surface F which appears in fig. 32 as the contour A C D $C_2 A_2$. Γ is the function symbol for the gamma function.

Let us consider now the immersion of a sharp knife-edge. $2\left(\frac{\pi}{2} - \beta\right) \to 0$ (fig. 33). It may be shown that, except for an infinitely small, insignificant region about C_2 the free surface at an infinitely flat slope and consequently the velocity field is as in fig. 32. For immersion at constant velocity V_o, as in para. 16. (impact force): $\frac{DB}{Dt} = \frac{2}{t} B$. With $V_o.t = T$ and with R for the expression in brackets in equation (82) [for $\beta \to \frac{\pi}{2}$, $R = 0{\cdot}89 \ldots \left(\frac{\pi}{2} - \beta\right)^{\cdot 2}$].

$$P = \frac{DB}{Dt} = 2\,\rho\,V_o^{\cdot 2}\,T \cot\beta \cdot R = 1{,}78\,\rho\,V_o^{\cdot 2}\,T\left(\frac{\pi}{2} - \beta\right)^{\cdot 2} \qquad (83).$$

The immersion also of a knife-edge with variable cutting angle may be determined by means of superposition from the foregoing. Further, as in para. 14, the impact of a knife-edge with given mass can be calculated.

Letter press to figs.

fig. 2. Zentrum = centre.

figs. 8.10. Ebene = plane $(z, \Gamma, \ln\frac{dz}{dw}, w)$

fig. 12. Bodenfläche = surface of bottom.

Druckfläche = pressure surface.

Spritzerwurzel = splash root.

Spritzen = splash.

Grenzbiereich = boundary region.

Hauptbereich = principal region.

Gleitkante = sliding edge.

Spritzerwurzel (" Vorderrand") = splash root ("leading edge").

fig. 13. Half space: The flow is uniquely determined with known V_n at the pressure surface and φ at the free surface. φ may be replaced by a known circulation - free velocity field $\mathfrak{W}_t$ at the free surface.

Flow with equivalent aerofoil. This flow in the simply connected space is uniquely determined with known velocity V_n at the surface of the aerofoil and with known discontinuities, the latter being present in the principal plane as surfaces of vorticity of the intensity $2\,[\mathfrak{n}_n\,\mathfrak{W}_t]$.

fig. 22. Spritzer = splash.

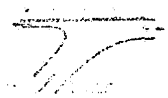
Plattenströmung = plate flow.

fig. 28. Erratum: $\mathfrak{W}_t$ should be $\mathfrak{W}t$.

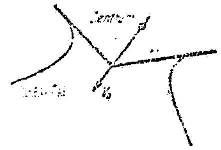
fig. 34. z, - h. Ebene = z, - h. plane.

fig. 21. z, w, $\ln \frac{dz}{dw}$, t, Ebene.

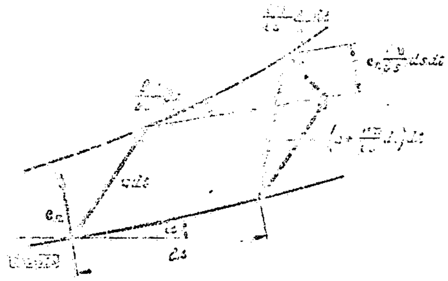
" " " " " plane.



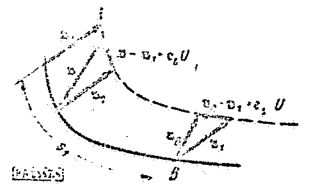
1.



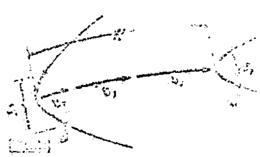
2.



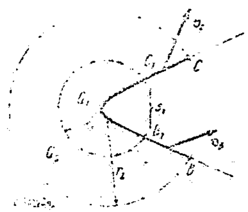
3.



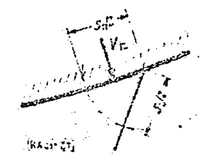
4.



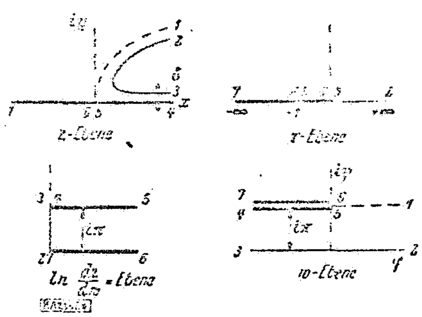
5.



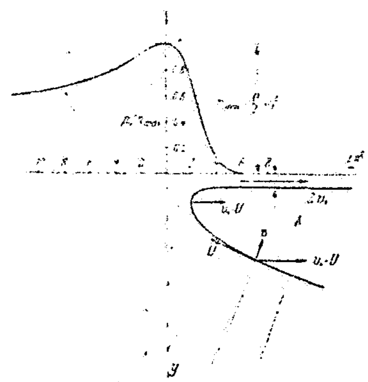
6.



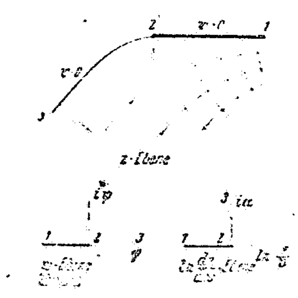
7.



8.

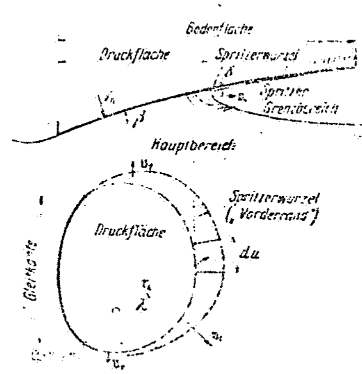


9.

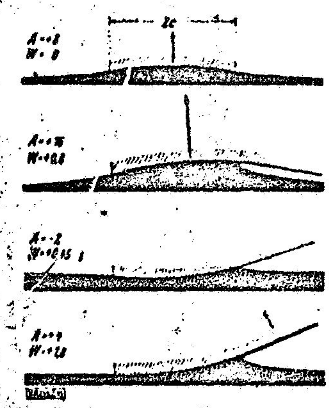


10.

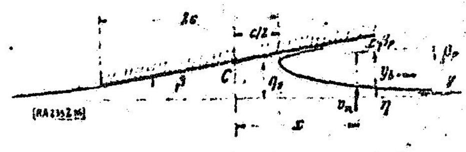
11.



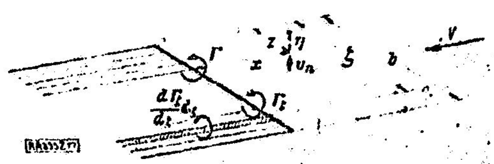
12.



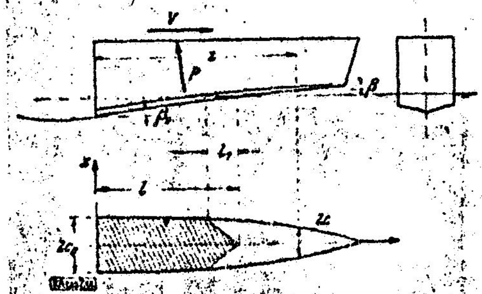
15.



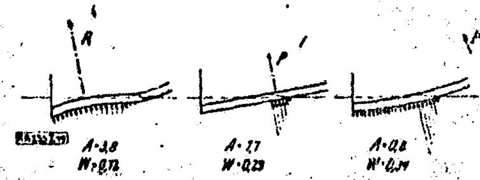
16.



17.



18.



19.

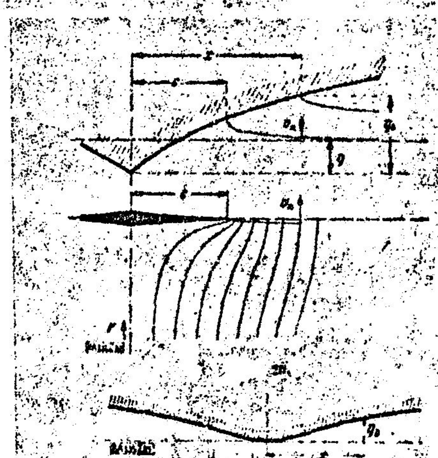
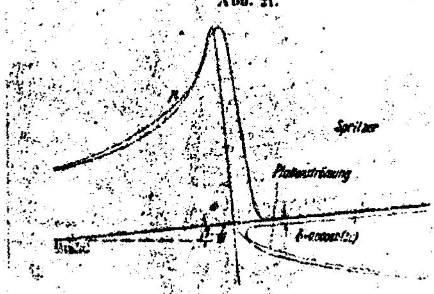
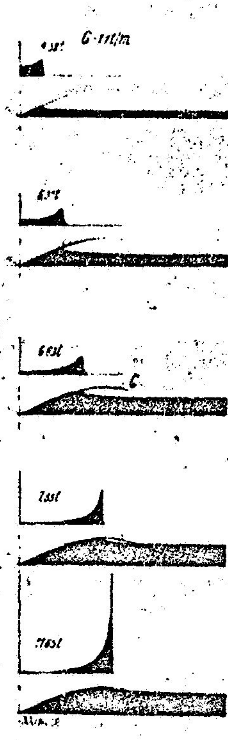


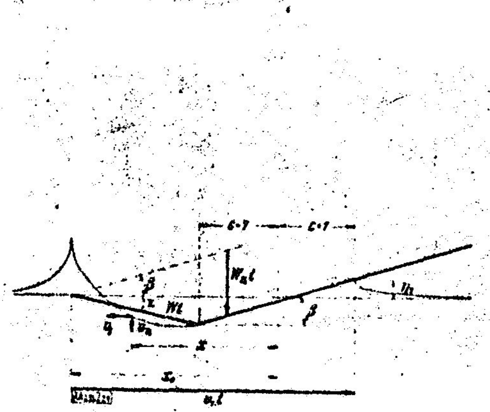
Abb. 21.

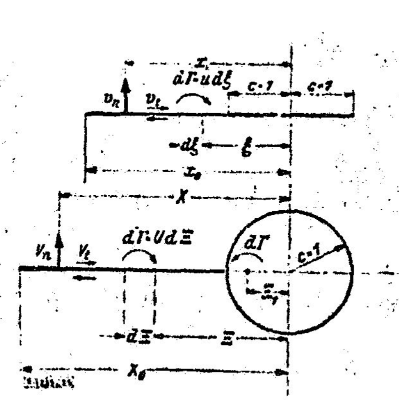


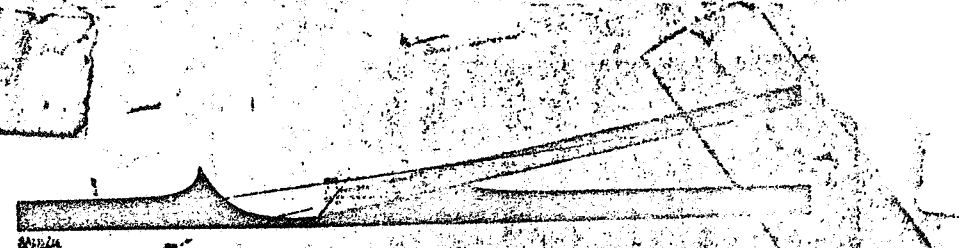
22.



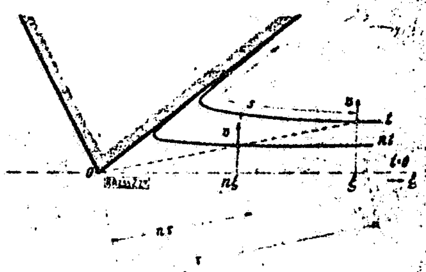
23.



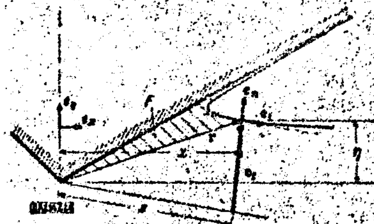




26.

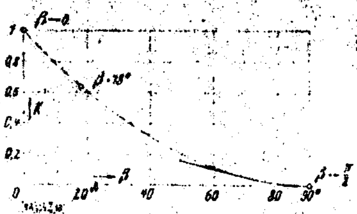


27.

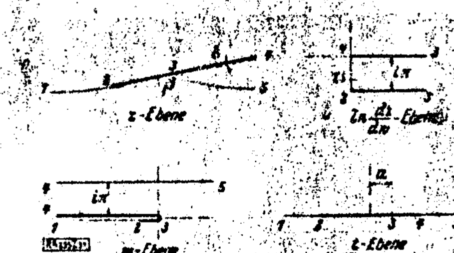


28.

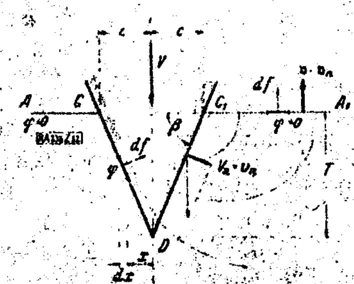
29.



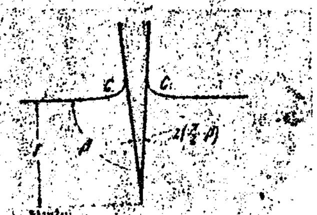
30.



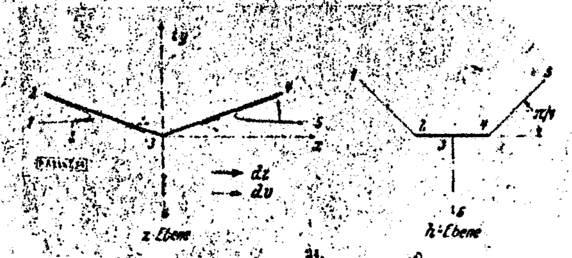
31.



32



33.



34.

VEHICLE RESEARCH CORPORATION

161 East California Boulevard

Pasadena, California

SYcamore 5-0491

27 January 1967

Library
National Aeronautics and Space Administration
Washington, D. C.

Gentlemen:

We would like to obtain a copy of the English translation of the following paper: Herbert Wagner "Uber Stoss - Gleitvorgange an der Oberflache von Flussigkeiten" published in No. 4, Vol. 12, August 1932 of "Zeitschrift Fur Angewandte Mathematik Und Mechanik, page 193-215.

A well-worn semi-legible copy of such a translation is available in the Caltech library, containing the stamp of NACA library, Langley Laboratory, and containing a number N-23507, with a note "translated in 1936," and with the title

"Phenomena Associated with Impacts and Sliding on Liquid Surfaces"

Your cooperation is appreciated.

Sincerely yours,

VEHICLE RESEARCH CORPORATION

Scott Rethorst
President

SR:mp